115AH Final Review Problems

Colin Ni

December 2, 2022

Let V and W be finite dimensional vector spaces over a field F, and let $T\colon V \to W$ be a linear map. Denote by V' the dual of V.

Straightforward problems

Problem 1. Let $U, W \leq V$ be subspaces such that $V = U + W$. Show that $V = U \oplus W$ if and only if $\dim V = \dim U + \dim W$.

Problem 2. Define what the transpose T^t of the map T is. Write a precise statement conveying the following idea: the matrix of a transpose is the transpose of the matrix. Prove your statement.

Problem 3. Suppose $F = \mathbb{C}$ and that the eigenvalues $\lambda_1, \dots, \lambda_n$ of T are distinct. Let $v_1, \dots, v_n$ be their corresponding eigenvectors. Is $v_1, \dots, v_n$ spanning? Is it linearly independent? Prove or disprove.

Problem 4. State Cauchy-Schwarz. State and prove the triangle inequality (for inner product spaces).

Problem 5. Suppose $F = \mathbb{C}$, let $\langle \cdot, \cdot \rangle$ be an inner product on V, and let S be a linear operator on V. Show that S is self adjoint if and only if $\langle Sv, v \rangle \in \mathbb{R}$ for all $v \in V$.

Fun problems

Problem 6. Suppose $F = \mathbb{R}$, and fix a list $v_1, \dots, v_m \in V$ of linearly independent vectors. How many lists $e_1, \dots, e_m \in V$ of orthonormal vectors are there such that

$$\operatorname{span}(v_1, \dots, v_j) = \operatorname{span}(e_1, \dots, e_j)$$

for all $j = 1, \dots, m$?

Problem 7. Suppose V and W have inner products $\langle \cdot, \cdot \rangle$. Show that for every functional $\varphi \in V'$, there exists a unique $u \in V$ such that

$$\varphi(\cdot) = \langle \cdot, u \rangle.$$

This is called the Riesz representation theorem. Use this to show that there exists a unique adjoint T^* of T, i.e. a linear map $T^*\colon W \to V$ such that

$$\langle Tv, w\rangle = \langle v, T^* w\rangle$$

for all $v \in V$ and $w \in W$.

Problem 8. The Fibonacci numbers are defined as

$$F_n = \begin{cases} F_{n-1} + F_{n-2} & n > 2 \\ 1 & n = 1 \text{ or } 2. \end{cases}$$

Show that

$$F_n = \frac{1}{\sqrt{5}}(\varphi^n - (\varphi - \sqrt{5})^n)$$

where

$$\varphi = \frac{1+\sqrt{5}}{2}$$

is the so-called golden ratio. *Hint: For any $a \in \mathbb{R}$, the eigenvectors of*

$$\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$$

are

$$\begin{pmatrix} \frac{1}{2}(a - \sqrt{a^2+4}) \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{1}{2}(a + \sqrt{a^2+4}) \\ 1 \end{pmatrix}.$$

Problem 9.

(a) Let X be a vector space over $\mathbb{C}$, let $T\colon X \to X$ be a linear operator, and fix $\lambda \in \mathbb{C}$. Show that the following statement is false: there exists $\epsilon > 0$ such that $T - zI$ is invertible whenever $|z - \lambda| < \epsilon$. *Hint: carefully write down the negation of the statement.*

(b) How can you strengthen the hypotheses to make the statement true?

Problem 10. Let $\varphi_1, \ldots, \varphi_n \in V'$ be linearly independent functionals, and let $v_1, \ldots, v_n \in V$ be such that $V = \ker \varphi_i \oplus \operatorname{span} v_i$. Prove or disprove: $v_1, \ldots, v_n$ are linearly independent.

Problem 11.

(a) Suppose V has an inner product $\langle \cdot, \cdot \rangle$, and let $U \leq V$ be a subspace. Give two definitions of orthogonal projection P_U onto U, one using orthogonal complements and one using orthonormal bases.

(b) Fix $v \in V$. Show that

$$\|v - P_U v\| \leq \|v - u\|$$

for all $u \in U$. *Hint: Draw a picture, and use the Pythagorean theorem for inner product spaces.* Show that this inequality is an equality if and only if $u = P_U v$.

(c) Fix $n \in \mathbb{N}$. Describe how to find a degree $\leq n$ polynomial p with real coefficients such that

$$\int_0^{2\pi} |\cos x - p(x)|^2 \, dx$$

is as small as possible.

Problem 12 (Extra fun, optional). Let

$$J_n(\lambda) = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$$

be an $n \times n$ matrix, where $\lambda \in \mathbb{C}$. This is called a Jordan block, and a matrix is said to be in Jordan normal form if it is a direct sum of Jordan blocks. Show that any operator on a finite-dimensional $\mathbb{C}$-vector space can be put uniquely (up to rearrangement of blocks) Jordan normal form.

Problem 13 (Optional). You may use Problem 12 for the following.

(a) Find the characteristic polynomial and the minimal polynomial of $J_n(\lambda)$.

(b) Find the characteristic polynomial and the minimal polynomial of a matrix in Jordan form. Show that there exists a matrix over $\mathbb{C}$ that has characteristic polynomial

$$x^3(x-3)^4(x+3)^4(x+i)^7$$

and minimal polynomial

$$x(x-3)^2(x+3)^3(x+i)^7.$$

(c) Suppose $F = \mathbb{C}$, and let S be a linear operator on V. Show that S is diagonalizable if and only if its minimal polynomial is a product of distinct linear factors. Show that V is not a direct sum of proper S-invariant subspaces if and only if the minimal polynomial of S is a linear factor with exponent $\dim V$.

(d) Let A be a linear operator on a 4-dimensional $\mathbb{C}$-vector space such that

$$A^4 + 2A^3 - 2A - I = 0.$$

Suppose $\dim \operatorname{im}(A + I) = 2$ and $|\operatorname{Tr} A| = 2$. What is the Jordan normal form of A?

Problem 14. Let S be a linear operator on V, and suppose that every dimension $\dim V - 1$ subspace of V is S-invariant. Show that S is a scalar multiple of the identity. *Hint: It suffices to show that every vector is an eigenvector of S.*

Problem 15. Let $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ be two inner products on V, and denote by $\|\cdot\|_1$ and $\|\cdot\|_2$ their induced norms. We say that the norms are equivalent if there exist positive numbers a, b such that

$$a\|v\|_1 \leq \|v\|_2 \leq b\|v\|_1$$

for all $v \in V$. Show that all inner products on $\mathbb{C}^n$ induce equivalent norms.

Problem 16 (Optional). Let S be a linear operator on V. Then S is called normal if $SS^* = S^*S$, where S^* denotes the adjoint of S (see Problem 7).

(a) Show that

$$A = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$$

is not self-adjoint but is normal.

(b) Show that S is normal if and only if $\|Sv\| = \|S^*v\|$ for all $v \in V$.

(c) Suppose S is normal. Show that S and S^* have the same eigenvectors. How are their eigenvalues related?

(d) Show that if S is normal, then eigenvectors of S corresponding to distinct eigenvectors are orthogonal.

(e) Show that if S is normal, then

$$\ker T^k = \ker T \quad \text{and} \quad \operatorname{im} T^k = \operatorname{im} T$$

for all positive integers k.